Journal of Combinatorial Theory, Series A 118 (2011) 949–961

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

On extensions of hyperplanes of dual polar spaces

Bart De Bruyn

Ghent University, Department of Mathematics, Krijgslaan 281 (S22), B-9000 Gent, Belgium

ARTICLE INFO

Article history:
Received 15 February 2010
Available online 19 June 2010

Keywords:
Dual polar space
Absolutely universal embedding
Minimal full polarized embedding
Grassmann embedding
(Extension of) hyperplanes

ABSTRACT

Let Δ be a thick dual polar space and F a convex subspace of diameter at least 2 of Δ. Every hyperplane G of the subgeometry $\widetilde{F}$ of Δ induced on F will give rise to a hyperplane H of Δ, the so-called extension of G. We show that F and G are in some sense uniquely determined by H. We also consider the following problem: if e is a full projective embedding of Δ and if e_F is the full embedding of $\widetilde{F}$ induced by e, does the fact that G arises from the embedding e_F imply that H arises from the embedding e? We will study this problem in the cases that e is an absolutely universal embedding, a minimal full polarized embedding or a Grassmann embedding of a symplectic dual polar space. Our study will allow us to prove that if e is absolutely universal, then also e_F is absolutely universal.

1. Introduction

In this paper, we make a study of a class of hyperplanes of dual polar spaces and use a property of these hyperplanes to obtain some structural information on the absolutely universal embedding of a fully embeddable thick dual polar space.

Suppose Δ is a dual polar space of rank n. If F is a convex subspace of diameter δ of Δ, then the points and lines of Δ which are contained in F define a point-line geometry $\widetilde{F}$ which is a dual polar space of rank δ. For every point x of Δ, there exists a unique point $\pi_F(x)$ in F nearest to x. Suppose G is a hyperplane of $\widetilde{F}$ and let H denote the set of all points of Δ at distance at most $n-\delta-1$ from F together with all points x of Δ at distance $n-\delta$ from F for which $\pi_F(x) \in G$. Then H is a hyperplane of Δ, called the *extension of* G. If $\delta < n$, then the extension is called *proper*. A hyperplane of Δ is called *reduced* if it is not the proper extension of some other hyperplane (of a convex subspace of Δ).

E-mail address: bdb@cage.ugent.be.

0097-3165/$ – see front matter
doi:10.1016/j.jcta.2010.06.003

The first main theorem of this paper states that every hyperplane of a not necessarily thick dual polar space Δ is uniquely expressible as the extension of a reduced hyperplane of a convex subspace of Δ.

Theorem 1.1. *Let Δ be a dual polar space of rank $n \geqslant 0$, let H be a hyperplane of Δ, let F_i, $i \in \{1,2\}$, be a convex subspace of Δ and let G_i, $i \in \{1,2\}$, be a reduced hyperplane of $\widetilde{F_i}$. If H is the extension of the hyperplane G_1 of $\widetilde{F_1}$ and the extension of the hyperplane G_2 of $\widetilde{F_2}$, then $F_1 = F_2$ and $G_1 = G_2$.*

From now on, we suppose that Δ is a thick dual polar space of rank $n \geqslant 2$ which is fully embeddable in a projective space. Then Δ admits the so-called absolutely universal embedding and the minimal full polarized embedding. Besides these two embeddings, there is another full projective embedding which will play a role in the main theorems of this paper, namely the Grassmann embedding of the symplectic dual polar space $DW(2n-1,\mathbb{F})$ where $\mathbb{F}$ is a field. A hyperplane H of Δ is said to *arise from a full projective embedding* $e : \Delta \to \Sigma$ of Δ if there exists a hyperplane α of Σ such that H consists of all points of Δ which are mapped by e into the hyperplane α.

In the literature, one can find plenty of constructions for hyperplanes of dual polar spaces. A question which arises after hyperplanes have been constructed is whether they arise from projective embeddings. The next theorem deals with the problem whether extensions of hyperplanes arise from embeddings (and which embeddings) if one knows that the original hyperplanes arise from a(n) (certain) embedding.

Theorem 1.2. *Let Δ be a fully embeddable thick dual polar space of rank $n \geqslant 2$, let F be a convex subspace of diameter $\delta \in \{2,\dots,n\}$ of Δ, let G be a hyperplane of $\widetilde{F}$ and let H be the hyperplane of Δ which extends the hyperplane G of $\widetilde{F}$.*

(1) *If G arises from the absolutely universal of $\widetilde{F}$, then H arises from the absolutely universal embedding of Δ.*
(2) *Suppose the projective space which affords the minimal full polarized embedding of $\widetilde{F}$ is finite-dimensional. If G arises from the minimal full polarized embedding of $\widetilde{F}$, then H arises from the minimal full polarized embedding of Δ.*
(3) *Suppose $\Delta \cong DW(2n-1,\mathbb{F})$ where $\mathbb{F}$ is some field. If G arises from the Grassmann embedding of $\widetilde{F} \cong DW(2\delta-1,\mathbb{F})$, then H arises from the Grassmann embedding of Δ.*

If a hyperplane of a fully embeddable thick dual polar space Δ of rank $n \geqslant 2$ arises from some full projective embedding, then it also arises from the absolutely universal embedding of Δ. So, Theorem 1.2(1) is equivalent with the following statement: "If G arises from some full projective embedding of $\widetilde{F}$, then H arises from some full projective embedding of Δ". If all hyperplanes of Δ arise from a given projective embedding e, then e necessarily is absolutely universal. The converse is false in general. It is possible that a hyperplane of Δ does not arise from its absolutely universal embedding.

Suppose $e : \Delta \to \Sigma$ is a full embedding of a thick dual polar space Δ and F is a convex subspace of diameter at least 2 of Δ. Then e will induce a full embedding e_F of $\widetilde{F}$ into a subspace Σ_F of Σ. An interesting problem is to determine which kind of embedding e_F is, for a given full projective embedding e of Δ. This problem has been solved in the case e is a minimal full polarized embedding [3, Theorem 1.6] or the Grassmann embedding of a symplectic dual polar space [3, Proposition 4.10].

Proposition 1.3. *(See [3].) Let Δ be a fully embeddable thick dual polar space of rank $n \geqslant 2$ and let F be a convex subspace of diameter $\delta \in \{2,\dots,n\}$ of Δ.*

(1) *If $e : \Delta \to \Sigma$ is the minimal full polarized embedding of Δ, then e_F is isomorphic to the minimal full polarized embedding of $\widetilde{F}$.*
(2) *If $\Delta \cong DW(2n-1,\mathbb{F})$ for some field $\mathbb{F}$ and $e : \Delta \to \Sigma$ is the Grassmann embedding of Δ, then e_F is isomorphic to the Grassmann embedding of $\widetilde{F} \cong DW(2\delta-1,\mathbb{F})$.*

The following theorem provides an answer to the above problem in the case that e is the absolutely universal embedding of Δ. Its proof will make use of Theorem 1.2(1).

Theorem 1.4. *Let Δ be a fully embeddable thick dual polar space of rank $n \geqslant 2$, let $\tilde{e}$ denote the absolutely universal embedding of Δ and let F be a convex subspace of diameter $\delta \in \{2, \dots, n\}$ of Δ. Then e_F is isomorphic to the absolutely universal embedding of $\widetilde{F}$.*

In the final section of this paper, we consider the following question for a full polarized projective embedding $e : \Delta \to \Sigma$ of a thick dual polar space Δ.

> Let F be a convex subspace of Δ, let G be a hyperplane of $\widetilde{F}$ and let H be the hyperplane of Δ obtained by extending G. Does the fact that G arises from e_F implies that H arises from e?

Theorem 1.2, Proposition 1.3 and Theorem 1.4 imply that the answer to the above question is affirmative if e is the absolutely universal embedding of Δ, the minimal full polarized embedding of Δ in case Σ_F is finite-dimensional or the Grassmann embedding of Δ in case Δ is isomorphic to a symplectic dual polar space. One might therefore wonder whether the answer is affirmative for any full polarized embedding of Δ. We will show that this is not the case by providing a class of counter examples.

2. Basic definitions and properties

Let Π be a polar space of rank $n \geqslant 1$ (Veldkamp [21]; Tits [20, Chapter 7]). With Π, there is associated a dual polar space Δ of rank n (Cameron [2]). This dual polar space Δ is the point-line geometry whose points are the maximal (i.e., $(n-1)$-dimensional) singular subspaces of Π and whose lines are the next-to-maximal (i.e., $(n-2)$-dimensional) singular subspaces of Π, with incidence being reverse containment. There exists a bijective correspondence between the set of nonempty convex subspaces of Δ and the set of possibly empty singular subspaces of Π. This correspondence is given as follows: if α is a singular subspace of dimension $n-1-\delta$, $\delta \in \{0, \dots, n\}$, of Π, then the set of all maximal singular subspaces of Π containing α is a convex subspace of diameter δ of Δ. The dual polar spaces of rank 1 are precisely the lines containing at least two points and the dual polar spaces of rank 2 are precisely the nondegenerate generalized quadrangles. By convention, a dual polar space of rank 0 is a point-line geometry which consists of one point (no lines).

Let Δ be a dual polar space of rank $n \geqslant 0$ with distance function $\mathrm{d}(\cdot,\cdot)$. The convex subspaces through a given point x of Δ define a projective space $\mathrm{Res}(x)$ of dimension $n-1$. The subspaces of dimension $i \in \{-1, 0, \dots, n-1\}$ of $\mathrm{Res}(x)$ correspond to the convex subspaces of diameter $i+1$ through x. If x is a point and F a convex subspace, then F contains a unique point $\pi_F(x)$ nearest to x and $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$ for every point y of F. We call $\pi_F(x)$ the *projection* of x onto F. A convex subspace of diameter δ of Δ is called a *quad* if $\delta = 2$ and a *max* if $\delta = n-1$. A dual polar space is called *thick* if each of its lines has at least three points and if for every quad Q and every point $x \in Q$, there are at least three lines through x contained in Q.

Let Δ be a dual polar space of rank $n \geqslant 0$ with point set $\mathcal{P}$. A set $H \neq \mathcal{P}$ of points of Δ is called a *hyperplane* if it intersects each line in either a singleton or the whole line. If x is a point of Δ, then the set H_x of all points at distance at most $n-1$ from x is a hyperplane of Δ, called the *singular hyperplane with deepest point* x. If $n = 0$, then x is the unique point of Δ and $H_x = \emptyset$. The deepest point of a singular hyperplane is uniquely determined by the hyperplane. Suppose F is a convex subspace of diameter $\delta \in \{0, \dots, n\}$ of Δ and G is a hyperplane of $\widetilde{F}$. If H denotes the set of all points at distance at most $n-\delta-1$ from F together with all points x at distance $n-\delta$ from F for which $\pi_F(x) \in G$, then H is a hyperplane of Δ, called the extension of G (De Bruyn and Vandecasteele [12, Proposition 1]). If $\delta = 0$, so F is a singleton $\{x\}$ and $G = \emptyset$, then H is the singular hyperplane of Δ with deepest point x.

Again, let Δ be a dual polar space. A *full (projective) embedding* of Δ is an injective mapping e from the point set $\mathcal{P}$ of Δ to the set of points of a projective space Σ satisfying the following two properties: (1) $\langle e(\mathcal{P})\rangle_\Sigma = \Sigma$; (2) e maps every line of Δ to some line of Σ. A dual polar space is called *fully embeddable* if it admits some projective embedding. Recall that if $e : \Delta \to \Sigma$ is a full embedding of Δ into a projective space Σ, then for every hyperplane α of Σ, $e^{-1}(e(\mathcal{P}) \cap \alpha)$ is a hyperplane of Δ. The hyperplane $e^{-1}(e(\mathcal{P}) \cap \alpha)$ is said to arise from e. A full embedding e of Δ is called *polarized* if every singular hyperplane of Δ arises from e. Two full embeddings $e_1 : \Delta \to \Sigma_1$ and $e_2 : \Delta \to \Sigma_2$ of Δ are called *isomorphic* ($e_1 \cong e_2$) if there exists an isomorphism $\phi : \Sigma_1 \to \Sigma_2$ such that $e_2 = \phi \circ e_1$. If $e : \Delta \to \Sigma$ is a full embedding of Δ and if α is a subspace of Σ satisfying

(C1) $\langle \alpha, e(x) \rangle_\Sigma \neq \alpha$ for every point x of Δ,
(C2) $\langle \alpha, e(x_1) \rangle_\Sigma \neq \langle \alpha, e(x_2) \rangle_\Sigma$ for any two distinct points x_1 and x_2 of Δ,

then there exists a full embedding e/α of Δ into the quotient space Σ/α mapping each point x of Δ to $\langle \alpha, e(x) \rangle_\Sigma$. If $e_1 : \Delta \to \Sigma_1$ and $e_2 : \Delta \to \Sigma_2$ are two full embeddings, then we say that $e_1 \geqslant e_2$ if there exists a subspace α of Σ_1 satisfying (C1), (C2) and $e_1/\alpha \cong e_2$.

If Δ is thick and fully embeddable into a projective space Σ, then by results of Kasikova and Shult [14, Section 4.6], Ronan [18, Proposition 3] and Tits [20, 8.6], there exists, up to isomorphism, a unique full embedding $\tilde{e} : \Delta \to \widetilde{\Sigma}$, such that $\tilde{e} \geqslant e$ for any full embedding e of Δ. (So, all full embeddings of Δ are defined over the same division ring.) The full embedding $\widetilde{e}$ is called the *absolutely universal embedding* of Δ. By Cardinali, De Bruyn and Pasini [4, Corollary 1.8], $\widetilde{e}$ is polarized. (For dual polar spaces of rank 2 or nondegenerate generalized quadrangles, this also follows from Johnson [13, Proposition 5.4].) If Δ is thick and fully embeddable, then by Cardinali, De Bruyn and Pasini [3, Theorem 1.4], there exists, up to isomorphism, a unique full polarized embedding $\bar{e}$ of Δ such that $e \geqslant \bar{e}$ for any full polarized embedding e of Δ. The embedding $\bar{e}$ is called the *minimal full polarized embedding* of Δ. If $e : \Delta \to \Sigma$ is a full polarized embedding of Δ and $\mathcal{P}$ is the point set of Δ, then also by [3], the subspace $R_e := \bigcap_{x \in \mathcal{P}} \langle e(H_x) \rangle_\Sigma$ satisfies the conditions (C1), (C2) and we have that $\bar{e} \cong e/R_e$. The subspace R_e of Σ is called the *nucleus* of e.

Suppose V is a vector space of dimension $2n \geqslant 4$ over a field $\mathbb{F}$ which is equipped with a nondegenerate alternating bilinear form $(\cdot,\cdot)$. The subspaces of V which are totally isotropic with respect to $(\cdot,\cdot)$ define a symplectic polar space $W(2n-1,\mathbb{F})$ and a symplectic dual polar space $DW(2n-1,\mathbb{F})$. If F is a convex subspace of diameter $\delta \geqslant 2$ of $DW(2n-1,\mathbb{F})$, then $\widetilde{F} \cong DW(2\delta-1,\mathbb{F})$. The function mapping each point $\langle \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \rangle$ of $DW(2n-1,\mathbb{F})$ to the point $\langle \bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_n \rangle$ of $\mathrm{PG}(\bigwedge^n V)$ defines a full embedding of $DW(2n-1,\mathbb{F})$ into a $[\binom{2n}{n} - \binom{2n}{n-2} - 1]$-dimensional subspace of $\mathrm{PG}(\bigwedge^n V)$. This embedding is called the *Grassmann embedding* of $DW(2n-1,\mathbb{F})$. The Grassmann embedding of $DW(2n-1,\mathbb{F})$ is polarized.

3. Proof of Theorem 1.1

Let Δ be a dual polar space of rank $n \geqslant 0$, let F be a convex subspace of diameter $\delta \in \{0,\ldots,n\}$ of Δ, let G be a hyperplane of $\widetilde{F}$ and let H denote the hyperplane of Δ which extends the hyperplane G of $\widetilde{F}$. If $*_1, *_2, \ldots, *_k$ are $k \geqslant 2$ objects of Δ (like points or nonempty sets of points), then $\langle *_1, *_2, \ldots, *_k \rangle$ denotes the smallest convex subspace of Δ containing $*_1, *_2, \ldots, *_k$.

Lemma 3.1. (1) *Let F' denote a convex subspace of Δ containing F, let G' denote the hyperplane of $\widetilde{F'}$ obtained by extending the hyperplane G of $\widetilde{F}$. Then the hyperplane H of Δ is the extension of the hyperplane G' of $\widetilde{F'}$.*

(2) *Suppose $\delta \neq n$. Let F' be a convex subspace of Δ which meets F and which contains a point at distance $n-\delta$ from F. If $F \cap F' \subseteq G$, then $F' \subseteq H$. If $F \cap F' \nsubseteq G$, then $F' \cap H$ is a hyperplane of $\widetilde{F'}$ which is the proper extension of the hyperplane $F \cap F' \cap G$ of $\widetilde{F \cap F'}$.*

Proof. (1) Let $\delta' \geqslant \delta$ be the diameter of F' and let x be an arbitrary point of Δ. Then $\mathrm{d}(x, \pi_F(x)) = \mathrm{d}(x, F) = \mathrm{d}(x, \pi_{F'}(x)) + \mathrm{d}(\pi_{F'}(x), F) = \mathrm{d}(x, F') + \mathrm{d}(\pi_{F'}(x), F)$ and $\pi_F(x) = \pi_F(\pi_{F'}(x))$.

(a) Suppose $\mathrm{d}(x, F') \leqslant n - \delta' - 1$. Since $\mathrm{d}(\pi_{F'}(x), F) \leqslant \delta' - \delta$, we have $\mathrm{d}(x, F) \leqslant n - \delta' - 1 + \delta' - \delta = n - \delta - 1$ and $x \in H$.

(b) Suppose $\mathrm{d}(x, F') = n - \delta'$ and $\mathrm{d}(\pi_{F'}(x), F) < \delta' - \delta$. Then $\pi_{F'}(x) \in G'$ and $x \in H$ since $\mathrm{d}(x, F) < n - \delta' + \delta' - \delta = n - \delta$.

(c) Suppose $\mathrm{d}(x, F') = n - \delta'$ and $\mathrm{d}(\pi_{F'}(x), F) = \delta' - \delta$. Then $\mathrm{d}(x, F) = n - \delta$. We have $x \in H \Leftrightarrow \pi_F(x) \in G \Leftrightarrow \pi_F(\pi_{F'}(x)) \in G \Leftrightarrow \pi_{F'}(x) \in G'$.

By (a), (b) and (c) above, H is the extension of the hyperplane G' of $\widetilde{F'}$.

(2) Let x be an arbitrary point of F' and let $y \in F \cap F'$. Since $\pi_F(x)$ lies on a shortest path between $x \in F'$ and $y \in F'$, $\pi_F(x) \in F \cap F'$. So, for every point x of F', $\pi_F(x) = \pi_{F \cap F'}(x)$ and $\mathrm{d}(x, F) = \mathrm{d}(x, F \cap F')$. Also, $\max\{\mathrm{d}(x, F \cap F') \mid x \in F'\} = \max\{\mathrm{d}(x, F) \mid x \in F'\} = n - \delta$. Claim (2) of the lemma immediately follows from these facts. $\square$

The maximal value that $\mathrm{d}(x, F)$ can attain if x ranges over all points of Δ is equal to $n - \delta$. A convex subspace F' of diameter δ is called *opposite* to F if every point of F' lies at maximal distance $n - \delta$ from F. If F' is opposite to F, then F is also opposite to F' and the map $F \to F'$; $x \mapsto \pi_{F'}(x)$ defines an isomorphism between $\widetilde{F}$ and $\widetilde{F'}$, with inverse map $F' \to F$, $x \mapsto \pi_F(x)$ (see e.g. [7, Theorem 1.10]).

Lemma 3.2. *Let x be a point at maximal distance $n - \delta$ from F and let F' denote a convex sub-2δ-gon through x for which $F' \cap \langle x, \pi_F(x)\rangle = \{x\}$. Then F' is opposite to F.*

Proof. By connectedness of F' and an inductive argument, it suffices to prove the following:

$(*)$ if $y \neq x$ is a point of F' collinear with x, then $\mathrm{d}(y, F) = n - \delta$ and $\langle y, \pi_F(y)\rangle \cap F' = \{y\}$.

Suppose $\mathrm{d}(y, F) \neq n - \delta$. Since $\mathrm{d}(y, F) \leqslant n - \delta$ and $\mathrm{d}(x, y) = 1$, we necessarily have $\mathrm{d}(y, F) = n - \delta - 1$. But then $\mathrm{d}(x, \pi_F(y)) \leqslant n - \delta$ and hence $\pi_F(x) = \pi_F(y)$. So, the point y which is on a shortest path between x and $\pi_F(x)$ must be contained in $F' \cap \langle x, \pi_F(x)\rangle = \{x\}$, a contradiction.

Hence, $\mathrm{d}(y, F) = n - \delta$. Since $y \notin \langle x, \pi_F(x)\rangle$, $\mathrm{d}(y, \pi_F(x)) = \mathrm{d}(y, x) + \mathrm{d}(x, \pi_F(x)) = n - \delta + 1$. Since $\langle x, \pi_F(x)\rangle \cap F' = \{x\}$ and $\langle x, \pi_F(x)\rangle$ is a max of $\widetilde{\langle y, \pi_F(x)\rangle}$, $\langle y, \pi_F(x)\rangle$ intersects F' in at most a line (look at $\mathrm{Res}(x)$). Hence, $\langle y, \pi_F(x)\rangle \cap F' = xy$. Since $\pi_F(y)$ is on a shortest path between y and $\pi_F(x)$, $\langle y, \pi_F(y)\rangle \cap F'$ must be either $\{y\}$ or xy. We prove that the latter possibility cannot occur. If the last possibility would occur, then the convex subspace $\langle y, \pi_F(y)\rangle$ of diameter $n - \delta$ would contain $\pi_F(x)$ since this point is contained on a shortest path between x and $\pi_F(y)$. This is however impossible since $\mathrm{d}(y, \pi_F(x)) = n - \delta + 1$. $\square$

Lemma 3.2 implies that every convex subspace admits an opposite convex subspace (of the same diameter).

Lemma 3.1(1) implies that the pair (F, G) is usually not uniquely determined by the hyperplane H. However, we can say the following:

Proposition 3.3. *Under the condition that G is a reduced hyperplane of $\widetilde{F}$, the convex subspace F of Δ and the hyperplane G of $\widetilde{F}$ are uniquely determined by H.*

Proof. The proof of the proposition will take place in a number of steps.

Claim 1. *Let F' be a convex subspace of diameter δ of Δ opposite to F. Then $H \cap F'$ is a reduced hyperplane of $\widetilde{F'}$.*

Proof. Since every point of F' lies at distance $n - \delta$ from F, we have $x \in H \cap F' \Leftrightarrow x \in F'$ and $\pi_F(x) \in G \Leftrightarrow x \in \pi_{F'}(G)$. So, $H \cap F' = \pi_{F'}(G)$. The claim then follows from the fact that the map $F \to F'$; $x \mapsto \pi_{F'}(x)$ defines an isomorphism between $\widetilde{F}$ and $\widetilde{F'}$. $\square$

Claim 2. *Let F' be a convex subspace of Δ containing a point x at distance at most $n - \delta - 1$ from F. Then $H \cap F'$ is either F' or a hyperplane of $\widetilde{F'}$ which is not reduced.*

Proof. Put $F'' := \langle x, F\rangle$. Let G'' denote the hyperplane of $\widetilde{F''}$ obtained by extending the hyperplane G of $\widetilde{F}$. Then by Lemma 3.1(1), the hyperplane H of Δ is the extension of the hyperplane G'' of $\widetilde{F''}$. The diameter of F'' is equal to $\delta'' := \mathrm{d}(x, F) + \delta \leqslant n - 1$. If F' does not contain points at distance $n - \delta''$

from F'', then $F' \subseteq H$. If F' contains points at distance $n-\delta''$ from F'', then by Lemma 3.1(2), $H \cap F'$ is either F' or a hyperplane of $\widetilde{F'}$ which is not reduced. $\square$

Claim 3. *Let F' be a convex subspace of Δ of diameter at least $\delta+1$. Then $H \cap F'$ is either F' or a hyperplane of $\widetilde{F'}$ which is not reduced.*

Proof. Let x be a point of F. Since the diameter of F' is at least $\delta+1$, $\mathrm{d}(x, F') \leqslant n-\delta-1$. So, F' contains a point at distance at most $n-\delta-1$ from F. The claim now follows from Claim 2. $\square$

Claim 4. *The number δ is uniquely determined by H.*

Proof. This is a corollary of Claims 1 and 3 and the fact that there exist convex subspaces of diameter δ opposite to F. $\square$

Claim 5. *Let $y \in F \setminus G$ and let F' be a convex subspace of diameter $n-\delta$ through y. If $|F' \cap F| \geqslant 2$, then $F' \subseteq H$. If $F' \cap F = \{y\}$, then $F' \cap H$ is the singular hyperplane of $\widetilde{F'}$ with deepest point y.*

Proof. Suppose $|F' \cap F| \geqslant 2$. Then $F' \cap F$ contains a line L. Since every point of F' has distance at most $n-\delta-1$ from some point of $L \subseteq F$, we have $F' \subseteq H$. Suppose $F' \cap F = \{y\}$. Let x be some point of F'. Since $\pi_F(x)$ is contained in some shortest path from $x \in F'$ to $y \in F'$, we have $\pi_F(x) \in F' \cap F = \{y\}$. Hence, $\pi_F(x) = y$ for every point $x \in F'$. Since $y \notin G$, this implies that a point $x \in F'$ belongs to H if and only if $\mathrm{d}(x, y) \leqslant n-\delta-1$. As a consequence, $F' \cap H$ is the singular hyperplane of $\widetilde{F'}$ with deepest point y. $\square$

Claim 6. *Let x be a point of Δ not belonging to H. Then there exists a (necessarily unique) convex subspace F_x of diameter $n-\delta$ through x such that: (i) $F_x \cap H$ is a singular hyperplane of $\widetilde{F_x}$; (ii) if F' is a convex subspace of diameter δ through x, then $F' \cap F_x = \{x\}$ if and only if $H \cap F'$ is a reduced hyperplane of $\widetilde{F'}$. Moreover, the deepest point of the singular hyperplane $F_x \cap H$ of $\widetilde{F_x}$ belongs to $F \setminus G$.*

Proof. Since $x \notin H$, $\mathrm{d}(x, F) = n-\delta$ and $\pi_F(x) \notin G$. Put $F_x := \langle x, \pi_F(x)\rangle$. Then F_x has diameter $n-\delta$. We prove that F_x satisfies the conditions (i) and (ii) of the claim. By Claim 5, $F_x \cap F = \{\pi_F(x)\}$ and $F_x \cap H$ is the singular hyperplane of $\widetilde{F_x}$ with deepest point $\pi_F(x) \in F \setminus G$.

Let F' be a convex subspace of diameter δ through x. If $F' \cap F_x = \{x\}$, then by Lemma 3.2 and Claim 1, $H \cap F'$ is a reduced hyperplane of $\widetilde{F'}$. Suppose $F' \cap F_x \neq \{x\}$. Then $F' \cap F_x$ contains a line L. This line L contains a point at distance $n-\delta-1$ from $\pi_F(x)$ and so we can apply Claim 2. Since $x \notin H$, $H \cap F'$ is a hyperplane of $\widetilde{F'}$ which is not reduced.

So, the convex subspace F_x satisfies the conditions of the claim. Let α be the $(n-\delta-1)$-dimensional subspace of $\mathrm{Res}(x)$ corresponding to F_x. By (ii), then $(\delta-1)$-dimensional subspaces of $\mathrm{Res}(x)$ disjoint from α are precisely those subspaces of $\mathrm{Res}(x)$ which correspond to a convex subspace F'' of diameter δ through x for which $F'' \cap H$ is a reduced hyperplane of $\widetilde{F''}$. Let $\mathcal{A}$ denote this set of $(\delta-1)$-dimensional subspaces of $\mathrm{Res}(x)$. If F' is a convex subspace of diameter $n-\delta$ through x satisfying the conditions (i) and (ii) above, then the $(n-\delta-1)$-dimensional subspace α' of $\mathrm{Res}(x)$ corresponding to F' is disjoint from each of the members of $\mathcal{A}$. Hence, $\alpha' = \alpha$, namely $F' = F_x$. This proves the uniqueness of F_x. $\square$

Now, for every point x of Δ not belonging to H, let $\theta(x)$ denote the unique deepest point of the singular hyperplane $H \cap F_x$ of $\widetilde{F_x}$.

Claim 7. *We have $\{\theta(x) \mid x \notin H\} = F \setminus G$.*

Proof. By Claim 6, $\{\theta(x) \mid x \notin H\} \subseteq F \setminus G$. Now, let y be an arbitrary point of $F \setminus G$ and let F' be a convex subspace of diameter $n-\delta$ through y for which $F' \cap F = \{y\}$. By Claim 5, $\theta(x) = y$ for every $x \in F'$ at distance $n-\delta$ from y. Hence, $\{\theta(x) \mid x \notin H\} = F \setminus G$. $\square$

By Claims 4, 6 and 7, the set $F \setminus G$ is uniquely determined by H. Now, fix a certain element $y \in F \setminus G$. Let $\mathcal{F}_y$ denote the set of all convex subspaces of diameter $n-\delta$ through y not contained in H. The set $\mathcal{F}_y$ is uniquely determined by H. By Claim 5, also F is uniquely determined by H: the lines through y contained in F are precisely the lines through y which are contained in none of the elements of $\mathcal{F}_y$ (look at $\mathrm{Res}(y)$). It follows that also $G = F \setminus (F \setminus G)$ is uniquely determined by H. □

Theorem 1.1 is precisely Proposition 3.3.

4. Proof of Theorem 1.2(1)

In order to prove Theorem 1.2(1), we need to recall some facts regarding simple connectedness of hyperplane complements of dual polar spaces.

Suppose H is a hyperplane of a thick dual polar space Δ of rank at least 3. Let Γ_1 be the graph whose vertices are the maxes of Δ which are not completely contained in H. Two distinct vertices M_1 and M_2 of Γ_1 are adjacent whenever $M_1 \cap M_2$ is not contained in H (so, $M_1 \cap M_2 \neq \emptyset$). A closed path $M_1, M_2, \ldots, M_k = M_1$ of Γ_1 is called *good* if $M_1 \cap M_2 \cap \cdots \cap M_k$ is not contained in H. Let Γ_2 be the graph whose vertices are the points of Δ not contained in H. Two distinct vertices of Γ_2 are adjacent whenever they are collinear as points of Δ. A closed path in Γ_2 is called *good* if there exists a max of Δ containing all its vertices. The complement $\Delta \setminus H$ of H in Δ is said to be *simply connected* if one of the following two equivalent conditions are satisfied:

- every closed path in Γ_1 decomposes into good closed paths;
- every closed path in Γ_2 decomposes into good closed paths.

For more background information on the topic of simple connectedness (of hyperplane complements of dual polar spaces), we refer to Pasini [17, Chapter 12] or Cardinali, De Bruyn and Pasini [4, Section 2]. The problem whether hyperplane complements of thick dual polar spaces are simply connected has been solved completely in two papers, one by Cardinali, De Bruyn and Pasini [4] and another one by McInroy and Shpectorov [16]. The following proposition can easily be extracted from these papers.

Proposition 4.1. *(See [4,16].) Let Δ be a thick dual polar space of rank at least 3, every line of which is incident with at least 4 points. If H is a hyperplane of Δ, then the complement $\Delta \setminus H$ of H in Δ is simply connected.*

In order to prove Theorem 1.2(1), we also need to invoke some results of Ronan. The following is a consequence of Corollary 4 of Ronan [18].

Proposition 4.2. *Let Δ be a fully embeddable thick dual polar space of rank $n \geqslant 3$ and let H be a hyperplane of Δ such that the complement of $\Delta \setminus H$ in Δ is simply connected. Suppose also that for every max M of Δ not contained in H, $M \cap H$ is a hyperplane of $\widetilde{M}$ which arises from the absolutely universal embedding of $\widetilde{M}$. Then H arises from the absolutely universal embedding of Δ.*

We are now ready to prove Theorem 1.2(1). The following proposition is a special case of Theorem 1.2(1), but it is equivalent to it in view of Lemma 3.1.

Proposition 4.3. *Let Δ be a fully embeddable thick dual polar space of rank $n \geqslant 3$ and let M be a max of Δ. Let G be a hyperplane of $\widetilde{M}$ which arises from the absolutely universal embedding of $\widetilde{M}$. Then the hyperplane H of Δ which extends the hyperplane G of $\widetilde{M}$ arises from the absolutely universal embedding of Δ.*

So, it suffices to prove Proposition 4.3. We will do this by induction on $n \geqslant 3$. So, suppose that Δ is a thick dual polar space of rank $n \geqslant 3$ and that Proposition 4.3 holds for all thick dual polar spaces Δ' of rank $n' \in \{3, \ldots, n-1\}$.

Suppose also that Δ is fully embeddable, that M is a max of Δ and that G is a hyperplane of $\widetilde{M}$ which arises from the absolutely universal embedding of $\widetilde{M}$. Let H be the hyperplane of Δ which extends the hyperplane G of $\widetilde{M}$. If every line of Δ contains precisely three points, then H arises from the absolutely universal embedding of Δ by Ronan [18, Corollary 2]. We will therefore suppose that every line of Δ is incident with at least 4 points. Then Proposition 4.1 implies that the complement $\Delta \setminus H$ of H in Δ is simply connected. By Proposition 4.2, it suffices to prove that for every max M' of Δ not contained in H, $M' \cap H$ is a hyperplane of $\widetilde{M'}$ which arises from the absolutely universal embedding of $\widetilde{M'}$.

Suppose first that M' is disjoint from M. Then $M' \cap H = \pi_{M'}(G)$. Since G arises from the absolutely universal embedding of $\widetilde{M}$ and the map $M \to M'; x \mapsto \pi_{M'}(x)$ defines an isomorphism between $\widetilde{M}$ and $\widetilde{M'}$, $\pi_{M'}(G)$ arises from the absolutely universal embedding of $\widetilde{M'}$.

Suppose next that M' meets M. Then $M \cap M'$ is a max of M'. If $M' \cap M$ is contained in G, then $M' \subseteq H$. Suppose therefore that $M' \cap M$ intersects G in a hyperplane U of $\widetilde{M' \cap M}$. Then $M' \cap H$ is a hyperplane of M' which is the extension of the hyperplane U of $\widetilde{M' \cap M}$. Suppose $n = 3$. Then M and M' are quads, $M \cap M'$ is a line, U is a point of $M \cap M'$ and $M' \cap H$ is a singular hyperplane of $\widetilde{M'}$. So, the hyperplane $M' \cap H$ of $\widetilde{M'}$ arises from the absolutely universal embedding of $\widetilde{M'}$ (which is polarized). Suppose therefore that $n \geqslant 4$. Since G arises from the absolutely universal embedding of $\widetilde{M}$, the hyperplane U of $\widetilde{M \cap M'}$ arises from some projective embedding of $\widetilde{M \cap M'}$ and hence also from the absolutely universal embedding of $\widetilde{M \cap M'}$. By the induction hypothesis, the hyperplane $H \cap M'$ of $\widetilde{M'}$ arises from the absolutely universal embedding of $\widetilde{M'}$.

We can now apply Proposition 4.2 and conclude that H must arise from the absolutely universal embedding of Δ.

5. Proof of Theorem 1.2(2)

Let Δ be a thick dual polar space of rank $n \geqslant 2$ with point set $\mathcal{P}$. Then every hyperplane of Δ is a maximal proper subspace of Δ by Blok and Brouwer [1, Theorem 7.3] or Shult [19, Lemma 6.1]. If H is a hyperplane of Δ arising from some full embedding $e : \Delta \to \Sigma$ of Δ, then since H is a maximal proper subspace of Δ, $\langle e(H)\rangle_\Sigma$ is a hyperplane of Σ and $H = e^{-1}(e(\mathcal{P}) \cap \langle e(H)\rangle_\Sigma)$. So, if H_1 and H_2 are two distinct hyperplanes of Δ arising from e, then the hyperplanes $\langle e(H_1)\rangle_\Sigma$ and $\langle e(H_2)\rangle_\Sigma$ of Σ are distinct. We then define $[H_1, H_2]_e$ as the set of all hyperplanes of the form $e^{-1}(e(\mathcal{P}) \cap A)$, where A is some hyperplane of Σ through $\langle e(H_1)\rangle_\Sigma \cap \langle e(H_2)\rangle_\Sigma$. Since H_1 and H_2 arise from e, they also arise from the absolutely universal embedding $\tilde{e}$ of Δ and we necessarily have $[H_1, H_2]_e = [H_1, H_2]$, where $[H_1, H_2] := [H_1, H_2]_{\tilde{e}}$. We also define $(H_1, H_2) := [H_1, H_2] \setminus \{H_1, H_2\}$.

In order to prove Theorem 1.2(2), we will make use of the following lemma. Notice that in the statement of this lemma the set $(\overline{G_1}, \overline{G_2})$ is well defined. Indeed, by Theorem 1.2(1) we know that $\overline{G_1}$ and $\overline{G_2}$ are hyperplanes arising from the absolutely universal embedding of Δ. Also, $\overline{G_1} \neq \overline{G_2}$ since $\overline{G_1} \cap F' = \pi_{F'}(G_1) \neq \pi_{F'}(G_2) = \overline{G_2} \cap F'$ for every convex subspace F' of diameter δ opposite to F (recall that the map $F \to F'; x \mapsto \pi_{F'}(x)$ defines an isomorphism between $\widetilde{F}$ and $\widetilde{F'}$).

Lemma 5.1. *Let Δ be a fully embeddable thick dual polar space of rank $n \geqslant 2$, let F be a convex subspace of diameter $\delta \geqslant 2$ of Δ and let G_1, G_2 be two distinct hyperplanes of $\widetilde{F}$ arising from the absolutely universal embedding of $\widetilde{F}$. For every hyperplane G of $\widetilde{F}$, let $\overline{G}$ denote the hyperplane of Δ which extends G. Then $(\overline{G_1}, \overline{G_2}) = \{\overline{G} \mid G \in (G_1, G_2)\}$.*

Proof. By Lemma 3.1 and a straightforward inductive argument, it suffices to prove the lemma in the case that F is a max of Δ. So, in the sequel we will indeed suppose that $n \geqslant 3$ and that F is a max of Δ.

We show that for every $H \in (\overline{G_1}, \overline{G_2})$, there exists a hyperplane G of $\widetilde{F}$ such that $H = \overline{G}$. Notice that $F \subseteq H$ since $F \subseteq \overline{G_1}$ and $F \subseteq \overline{G_2}$. Let x be an arbitrary point of F. We show that either $x^\perp \subseteq H$ or $x^\perp \cap H = x^\perp \cap F$, where $x^\perp$ denotes the set of points at distance at most 1 from x. If this were not the case, then there would exist two distinct lines L_1 and L_2 through x not contained in F such that $L_1 \subseteq H$ and $L_2 \nsubseteq H$. Let Q denote the unique quad through L_1 and L_2 and let L_3 be the line $Q \cap F$. Now, $Q \cap H$ is a hyperplane of $\widetilde{Q}$ which is necessarily a proper subquadrangle of $\widetilde{Q}$ since $L_1, L_3 \subseteq H$ and $L_2 \nsubseteq H$. Let y denote a point of $L_3 \cap G_1$ and let L_4 be a line of $Q \cap H$ through y distinct from L_3. Since L_4 is contained in $H \in (\overline{G_1}, \overline{G_2})$ and $\overline{G_1}$, it is also contained in $\overline{G_2}$. So, $y \in G_2$. Now, since $y^\perp \subseteq \overline{G_2} \cap \overline{G_1}$, $y^\perp \cap Q$ would be contained in $Q \cap H$, in contradiction with the fact that $Q \cap H$ is a proper subquadrangle of $\widetilde{Q}$. Hence, for every $x \in F$, $x^\perp \cap H$ is either $x^\perp$ or $x^\perp \cap F$. Now, let G denote the set of all $x \in F$ for which $x^\perp \cap H = x^\perp$. Then $H = F \cup (\bigcup_{x \in G} x^\perp)$. Let F' be a max of Δ disjoint from F. Since $F' \cap H = \bigcup_{x \in G}(x^\perp \cap F') = \pi_{F'}(G)$ is a hyperplane of $\widetilde{F'}$, G is a hyperplane of $\widetilde{F}$ and $H = F \cup (\bigcup_{x \in G} x^\perp) = \overline{G}$. Since $G = \pi_F(H \cap F')$, we have

$$H = \overline{\pi_F(H \cap F')}. \tag{1}$$

Now, let $\tilde{e} : \Delta \to \widetilde{\Sigma}$ denote the absolutely universal embedding of Δ. Put $\Sigma_{F'} := \langle \tilde{e}(F') \rangle$. By Theorem 1.2(1), the hyperplane $\overline{G_i}$, $i \in \{1,2\}$, arises from $\tilde{e}$. Hence, $A_i := \langle \tilde{e}(\overline{G_i}) \rangle$ is a hyperplane of $\widetilde{\Sigma}$. Since G_1 and G_2 are distinct, $\overline{G_1}$ and $\overline{G_2}$ are distinct and hence also A_1 and A_2 are distinct. The hyperplanes of $\widetilde{\Sigma}$ through $A_1 \cap A_2$ define a line in the dual space $\widetilde{\Sigma}^*$ of $\widetilde{\Sigma}$. Since $\overline{G_1} \cap F' = \pi_{F'}(G_1) \neq \pi_{F'}(G_2) = \overline{G_2} \cap F'$, the subspaces of $\Sigma_{F'}$ of the form $A \cap \Sigma_{F'}$, where A is some hyperplane of $\widetilde{\Sigma}$ through $A_1 \cap A_2$ is a line of the dual space $\Sigma_{F'}^*$ of $\Sigma_{F'}$. From this, it follows that

$$\left(F' \cap \overline{G_1}, F' \cap \overline{G_2}\right) = \left\{F' \cap H \;\middle|\; H \in (\overline{G_1}, \overline{G_2})\right\}. \tag{2}$$

Clearly,

$$\left(F' \cap \overline{G_1}, F' \cap \overline{G_2}\right) = \left(\pi_{F'}(G_1), \pi_{F'}(G_2)\right). \tag{3}$$

Since the map $F \to F'; x \mapsto \pi_{F'}(x)$ defines an isomorphism between $\widetilde{F}$ and $\widetilde{F'}$, we have

$$\left(\pi_{F'}(G_1), \pi_{F'}(G_2)\right) = \left\{\pi_{F'}(G) \;\middle|\; G \in (G_1, G_2)\right\}. \tag{4}$$

By (2), (3) and (4),

$$\left\{F' \cap H \;\middle|\; H \in (\overline{G_1}, \overline{G_2})\right\} = \left\{\pi_{F'}(G) \;\middle|\; G \in (G_1, G_2)\right\}. \tag{5}$$

By Eqs. (1) and (5), $(\overline{G_1}, \overline{G_2}) = \{H \mid H \in (\overline{G_1}, \overline{G_2})\} = \{\overline{\pi_F(H \cap F')} \mid H \in (\overline{G_1}, \overline{G_2})\} = \{\overline{\pi_F(\pi_{F'}(G))} \mid G \in (G_1, G_2)\} = \{\overline{G} \mid G \in (G_1, G_2)\}$. $\square$

In order to prove Theorem 1.2(2), we will also make use of the following lemma. Observe that the sets (H_{i-1}, G_i) which occur in the statement of this lemma are well-defined by an inductive argument and the fact that the singular hyperplanes of Δ arise from the absolutely universal embedding of Δ.

Lemma 5.2. *Let Δ be a fully embeddable thick dual polar space of rank $n \geqslant 2$ and suppose the minimal full polarized embedding $\bar{e}$ of Δ is finite-dimensional. Then a hyperplane H of Δ arises from $\bar{e}$ if and only if there exists a $k \geqslant 1$ and hyperplanes $H_1, H_2, \ldots, H_k$ of Δ satisfying*:

(1) *H_1 is a singular hyperplane of Δ*;
(2) *for every $i \in \{2, \ldots, k\}$, $H_i \in (H_{i-1}, G_i)$ for some singular hyperplane G_i of Δ distinct from H_{i-1}*;
(3) *$H_k = H$.*

Proof. Let $\tilde{e} : \Delta \to \widetilde{\Sigma}$ denote the absolutely universal embedding of Δ and let $\mathcal{P}$ denote the point set of Δ. Recall that $\bar{e} \cong \tilde{e}/R_{\tilde{e}}$, where $R_{\tilde{e}}$ is the nucleus of $\tilde{e}$. The conditions of the lemma imply that $R_{\tilde{e}}$ has finite co-dimension in $\widetilde{\Sigma}$. So, there exists an $l \in \mathbb{N} \setminus \{0\}$ and points $x_1, \ldots, x_l$ of Δ such that $R_{\tilde{e}} = \bigcap_{1 \leqslant i \leqslant l} A_i$, where $A_i = \langle \tilde{e}(H_{x_i}) \rangle$, $i \in \{1, \ldots, l\}$.

Suppose $H_1, H_2, \ldots, H_k$ is a set of $k \geqslant 1$ hyperplanes of Δ satisfying the conditions (1), (2) and (3) of the lemma. By induction on $i \in \{1, \ldots, k\}$, we immediately see that each H_i, $i \in \{1, \ldots, k\}$, arises from $\bar{e}$.

Conversely, suppose that H is a hyperplane of Δ arising from $\bar{e}$. Put $A = \langle \tilde{e}(H) \rangle$. Then $A \in \langle A_1, \ldots, A_l \rangle^*$ in the dual space $\widetilde{\Sigma}^*$ of $\widetilde{\Sigma}$. Let k be the smallest nonnegative integer such that there exists a set of k hyperplanes of $\{A_1, \ldots, A_l\}$ generating a subspace of $\widetilde{\Sigma}^*$ containing A. Without loss of generality, one may suppose that $A \in \langle A_1, \ldots, A_k\rangle^*$. Put $H_1 := \tilde{e}^{-1}(\tilde{e}(\mathcal{P}) \cap A_1)$ and $G_i := \tilde{e}^{-1}(\tilde{e}(\mathcal{P}) \cap A_i)$ for every $i \in \{2, \ldots, k\}$. Since $A \in \langle A_1, \ldots, A_k\rangle^*$, there exist hyperplanes $H_2, \ldots, H_k$ of Δ such that $H_k = H$ and $H_i \in (H_{i-1}, G_i)$ for every $i \in \{2, \ldots, k\}$. □

Definition. Let Δ and $\bar{e}$ be as in Lemma 5.2. If H is a hyperplane of Δ arising from $\bar{e}$, then the smallest nonnegative integer k for which there exist hyperplanes $H_1, H_2, \ldots, H_k$ of Δ satisfying the conditions (1), (2) and (3) of Lemma 5.2 is called the *index of* H.

We are now ready to prove Theorem 1.2(2). So, let Δ be a fully embeddable thick dual polar space of rank $n \geqslant 2$, let F be a convex subspace of diameter $\delta \in \{2, \ldots, n\}$ of Δ and suppose the projective space which affords the minimal full polarized embedding $\bar{e}_F$ of $\widetilde{F}$ is finite-dimensional. Let G be a hyperplane of $\widetilde{F}$ arising from $\bar{e}_F$ and let i be the *index* of G. We shall prove by induction on i that the hyperplane H of Δ which extends the hyperplane G of $\widetilde{F}$ arises from the minimal full polarized embedding $\bar{e}$ of Δ.

Suppose first that $i = 1$. Then G is a singular hyperplane of $\widetilde{F}$ and H is a singular hyperplane of Δ. Hence, H arises from $\bar{e}$.

Suppose $i \geqslant 2$. Then $H \in (H_1, H_2)$, where H_1 is a hyperplane of index $i-1$ of $\widetilde{F}$ arising from $\bar{e}_F$ and H_2 is a singular hyperplane distinct from H_1. By the induction hypothesis, $\overline{H_1}$ and $\overline{H_2}$ arise from $\bar{e}$. Hence, by Lemma 5.1, also $\overline{H} \in (\overline{H_1}, \overline{H_2})$ arises from $\bar{e}$.

6. Proof of Theorem 1.2(3)

Let $n \in \mathbb{N} \setminus \{0, 1\}$ and let $\mathbb{F}$ be a field.

The following proposition is a special case of Theorem 1.2(3), but is equivalent to it in view of Lemma 3.1.

Proposition 6.1. *Let M be a max of the dual polar space $DW(2n-1, \mathbb{F})$, $n \geqslant 3$. Let G be a hyperplane of $\widetilde{M}$ and let H be the hyperplane of Δ which extends the hyperplane G of $\widetilde{M}$. If G arises from the Grassmann embedding of $\widetilde{M}$, then H arises from the Grassmann embedding of Δ.*

So, it suffices to prove Proposition 6.1. Suppose M is a max of $DW(2n-1, \mathbb{F})$, where $n \geqslant 3$, let G be a hyperplane of $\widetilde{M}$ arising from the Grassmann embedding of $\widetilde{M}$ and let H be the hyperplane of $DW(2n-1, \mathbb{F})$ which extends the hyperplane G of $\widetilde{M}$. We need to prove that H arises from the Grassmann embedding of Δ.

If $\mathbb{F}$ is not isomorphic to $\mathbb{F}_2$, then by Cooperstein [6, Theorem B], De Bruyn and Pasini [11, Corollary 1.2] and Kasikova and Shult [14, Section 4.6], the Grassmann embedding of $DW(2n-1, \mathbb{F})$ is absolutely universal. By Theorem 1.2(1), we then know that H arises from the Grassmann embedding of Δ. So, it remains to show that H arises from the Grassmann embedding of Δ in the special case that $\mathbb{F}$ is isomorphic to $\mathbb{F}_2$. In fact, the reasoning which we will give below works for any field $\mathbb{F}$ which admits a quadratic Galois extension $\mathbb{F}'$. So, let $\mathbb{F}$ and $\mathbb{F}'$ be like that and let θ be the unique nontrivial element in the Galois group $\mathrm{Gal}(\mathbb{F}'/\mathbb{F})$.

Consider in $\mathrm{PG}(2n-1, \mathbb{F}')$ a Hermitian variety $\mathcal{H}$ whose equation with respect to a suitable reference system is given by $(X_1X_2^\theta - X_2X_1^\theta) + \cdots + (X_{2n-1}X_{2n}^\theta - X_{2n}X_{2n-1}^\theta) = 0$. With this Hermitian variety $\mathcal{H}$ there is associated a Hermitian polar space $H(2n-1, \mathbb{F}'/\mathbb{F})$ and a Hermitian dual polar space $DH(2n-1, \mathbb{F}'/\mathbb{F})$. The dual polar space $DW(2n-1, \mathbb{F})$ can be isometrically embedded as a full subgeometry in $DH(2n-1, \mathbb{F}'/\mathbb{F})$, see De Bruyn [9]. The dual polar space $DH(2n-1, \mathbb{F}'/\mathbb{F})$ admits a full embedding e_1 into $\Sigma_1 = \mathrm{PG}(\binom{2n}{n} - 1, \mathbb{F})$ which is called the *Grassmann embedding* of $DH(2n-1, \mathbb{F}'/\mathbb{F})$, see Cooperstein [5] and De Bruyn [8]. The embedding e_1 induces an embedding e_2 of $DW(2n-1, \mathbb{F})$ into a subspace Σ_2 of Σ_1 which is isomorphic to the Grassmann embedding of $DW(2n-1, \mathbb{F})$, see [9, Theorem 1.1].

Lemma 6.2. *If K_2 is a hyperplane of $DW(2n-1,\mathbb{F})$ arising from the Grassmann embedding of $DW(2n-1,\mathbb{F})$. Then there exists a hyperplane K_1 of $DH(2n-1,\mathbb{F}'/\mathbb{F})$ arising from the Grassmann embedding of $DH(2n-1,\mathbb{F}'/\mathbb{F})$ such that $K_2 = K_1 \cap \mathcal{P}_2$, where $\mathcal{P}_2$ denotes the point set of $DW(2n-1,\mathbb{F})$.*

Proof. Let A_2 be the hyperplane of Σ_2 such that $K_2 = e_2^{-1}(e_2(\mathcal{P}_2)\cap A_2)$, let A_1 be a hyperplane of Σ_1 intersecting Σ_2 in A_2 and put $K_1 := e_1^{-1}(e_1(\mathcal{P}_1)\cap A_1)$, where $\mathcal{P}_1$ denotes the point set of $DH(2n-1,\mathbb{F}'/\mathbb{F})$. Then $K_2 = K_1 \cap \mathcal{P}_2$. □

Now, the max M of $DW(2n-1,\mathbb{F})$ is contained in a unique max M' of $DH(2n-1,\mathbb{F}'/\mathbb{F})$. Observe that $\widetilde{M} \cong DW(2n-3,\mathbb{F})$ and $\widetilde{M'} \cong DH(2n-3,\mathbb{F}'/\mathbb{F})$. Moreover, the inclusion of M into M' defines a full isometric embedding of $\widetilde{M}$ into $\widetilde{M'}$. By Lemma 6.2, there exists a hyperplane G' of $\widetilde{M'}$ arising from the Grassmann embedding of $\widetilde{M'}$ such that $G = G' \cap M$. Now, let H' be the hyperplane of $DH(2n-1,\mathbb{F}'/\mathbb{F})$ which extends the hyperplane G' of $\widetilde{M'}$. Then we have $H' \cap \mathcal{P}_2 = H$. By Cardinali, De Bruyn and Pasini [3, Theorem 4.1], the Grassmann embedding of $DH(2n-1,\mathbb{F}'/\mathbb{F})$ is isomorphic to the minimal full polarized embedding of $DH(2n-1,\mathbb{F}'/\mathbb{F})$ (the finiteness assumption in [3] is not essential). So, by Theorem 1.2(2), H' arises from the Grassmann embedding of $DH(2n-1,\mathbb{F}'/\mathbb{F})$. So, there exists a hyperplane B_1 of Σ_1 such that $H' = e_1^{-1}(e_1(\mathcal{P}_1)\cap B_1)$. Put $B_2 = B_1 \cap \Sigma_2$. Since $H = H' \cap \mathcal{P}_2$, $H = e_2^{-1}(e_2(\mathcal{P}_2)\cap B_2)$. Hence, H arises from the Grassmann embedding of $DW(2n-1,\mathbb{F})$, as we needed to prove.

7. Proof of Theorem 1.4

Lemma 7.1. *Let Δ be a fully embeddable thick dual polar space of rank $n \geqslant 2$. Let e be a full embedding of Δ which is not isomorphic to the absolutely universal embedding $\tilde{e}$ of Δ. Then there exists a hyperplane H of Δ which arises from $\tilde{e}$, but not from e.*

Proof. Let $\widetilde{\Sigma}$ be the projective space which affords the absolutely universal embedding $\tilde{e}$. Then there exists a nonempty subspace U of $\widetilde{\Sigma}$ satisfying the conditions (C1) and (C2) of Section 2 such that $\tilde{e}/U \cong e$. Now, let A be a hyperplane of $\widetilde{\Sigma}$ not containing U and let H be the hyperplane $\tilde{e}^{-1}(\tilde{e}(\mathcal{P})\cap A)$ of Δ, where $\mathcal{P}$ denotes the point set of Δ. Then H arises from $\tilde{e}$, but not from e. □

Now, let Δ be a fully embeddable thick dual polar space of rank $n \geqslant 2$ with point set $\mathcal{P}$, let $\tilde{e}: \Delta \to \widetilde{\Sigma}$ denote the absolutely universal embedding of Δ and let F be a convex subspace of diameter $\delta \in \{2,\dots,n\}$ of Δ. We will prove that the embedding $e_F : \widetilde{F} \to \Sigma_F$ of $\widetilde{F}$ induced by $\tilde{e}$ is isomorphic to the absolutely universal embedding $\widetilde{e_F}$ of $\widetilde{F}$. It suffices to prove this in the case that F is a max of Δ (otherwise apply a straightforward induction).

Suppose e_F is not isomorphic to the absolutely universal embedding of $\widetilde{F}$. By Lemma 7.1, there exists a hyperplane G of $\widetilde{F}$ which arises from $\widetilde{e_F}$, but not from e_F. Let F' be a max disjoint from F and put $G' := \pi_{F'}(G)$. Since the map $F \to F'; x \mapsto \pi_{F'}(x)$ defines an isomorphism between $\widetilde{F}$ and $\widetilde{F'}$, the hyperplane G' of $\widetilde{F'}$ arises from the absolutely universal embedding $\widetilde{e_{F'}}$ of $\widetilde{F'}$. Now, let H be the hyperplane of Δ which extends the hyperplane G' of $\widetilde{F'}$. By Theorem 1.2(1), H arises from the absolutely universal embedding $\tilde{e}$ of Δ. So, there exists a hyperplane A of $\widetilde{\Sigma}$ such that $H = \tilde{e}^{-1}(\tilde{e}(\mathcal{P})\cap A)$. Since $G = H \cap F$, we have $G = \tilde{e}^{-1}(\tilde{e}(F)\cap(A\cap\Sigma_F))$, i.e. $G = e_F^{-1}(e_F(F)\cap(A\cap\Sigma_F))$. So, G arises from e_F, a contradiction. Hence, e_F must be isomorphic to the absolutely universal embedding of $\widetilde{F}$.

8. Construction of some special embeddings

Consider the following question for a full polarized embedding $e : \Delta \to \Sigma$ of a thick dual polar space Δ of rank $n \geqslant 2$.

> Let F be a convex subspace of Δ, let G be a hyperplane of $\widetilde{F}$ and let H be the hyperplane of Δ obtained by extending G. Does the fact that G arises from e_F implies that H arises from e?

As explained in Section 1, our main results imply that the answer to the above question is affirmative if e is the absolutely universal embedding of Δ, the minimal full polarized embedding of Δ in case Σ_F is finite-dimensional or the Grassmann embedding of Δ in case Δ is isomorphic to a symplectic dual polar space. One might therefore wonder whether the answer is affirmative for any full polarized embedding of Δ. We show that this is not the case. A source for counter examples will be provided in Proposition 8.2 below. We will need the following lemma.

Lemma 8.1. *Let $e : \Delta \to \Sigma$ be a full polarized embedding of a thick dual polar space Δ of rank $n \geqslant 2$ and let F be a convex subspace of diameter $\delta \geqslant 2$ of Δ. Then e_F is polarized and $R_{e_F} \subseteq R_e$.*

Proof. The fact that e_F is polarized was proved in Cardinali, De Bruyn and Pasini [3, Theorem 1.5].

For every point y of F, let H'_y denote the singular hyperplane of $\widetilde{F}$ with deepest point y. For every point y of F, there exists a point x at distance $n-\delta$ from F such that $y = \pi_F(x)$; for such a point x, there holds that $H'_y \subseteq H_x$ and $\langle e_F(H'_y)\rangle \subseteq \langle e(H_x)\rangle$. If x is a point at distance at most $n-\delta-1$ from F, then $F \subseteq H_x$ and $\langle e_F(F)\rangle \subseteq \langle e(H_x)\rangle$. It follows that $R_{e_F} = \bigcap_{y\in F}\langle e(H'_y)\rangle \subseteq \bigcap_{x\in\mathcal{P}}\langle e(H_x)\rangle = R_e$. Here, $\mathcal{P}$ denotes the point set of Δ. $\square$

Proposition 8.2. *Let Δ be a fully embeddable thick dual polar space of rank $n \geqslant 3$ and let F_1 be a convex subspace of diameter $\delta \in \{2,\dots,n-1\}$ of Δ. Suppose the absolutely universal embedding and the minimal full polarized embedding of $\widetilde{F_1}$ are not isomorphic. Then there exists a full polarized embedding e of Δ and a hyperplane G of $\widetilde{F_1}$ such that*:

(1) *G arises from e_{F_1};*
(2) *the hyperplane H of Δ which extends the hyperplane G of $\widetilde{F_1}$ does not arise from e.*

Proof. Let $\tilde{e} : \Delta \to \widetilde{\Sigma}$ denote the absolutely universal embedding of Δ. Let F_2 be a convex subspace of diameter δ opposite to F_1. Since $\delta < n$, F_1 and F_2 are disjoint. By Theorem 1.4, $\tilde{e}_i := \tilde{e}_{F_i}$, $i \in \{1,2\}$, is isomorphic to the absolutely universal embedding of $\widetilde{F_i}$. Put $R := R_{\tilde{e}}$ and $R_i := R_{\tilde{e}_i}$, $i \in \{1,2\}$. By Lemma 8.1, $\langle R_1, R_2\rangle \subseteq R$. Let $\bar{e}_i$, $i \in \{1,2\}$, denote the minimal full polarized embedding of $\widetilde{F_i}$. Since $\bar{e}_1$ and $\tilde{e}_1$ are not isomorphic, $R_1 \neq \emptyset$ and there exists a hyperplane G of $\widetilde{F_1}$ which arises from $\tilde{e}_1$ but not from $\bar{e}_1$ (recall Lemma 7.1). Let H denote the hyperplane of Δ which extends the hyperplane G of $\widetilde{F_1}$. By Theorem 1.2(1), H arises from $\tilde{e}$. So, there exists a hyperplane A of $\widetilde{\Sigma}$ such that $H = \tilde{e}^{-1}(\tilde{e}(\mathcal{P}) \cap A)$, where $\mathcal{P}$ denotes the point set of Δ. Since $F_1 \subseteq H$, $R_1 \subseteq A$.

Since the map $F_1 \to F_2; x \mapsto \pi_{F_2}(x)$ defines an isomorphism between $\widetilde{F_1}$ and $\widetilde{F_2}$, $H \cap F_2 = \pi_{F_2}(G)$ is a hyperplane of $\widetilde{F_2}$ arising from $\widetilde{e_2}$ but not from $\bar{e}_2$. So, $A \cap R_2$ is a hyperplane of R_2. Since $\langle R_1, R_2\rangle \subseteq R$, $A \cap R$ is a hyperplane of R containing R_1. Since G does not arise from $\bar{e}_1$, $\langle \widetilde{e_1}(G)\rangle$ intersects R_1 in a hyperplane α of R_1. Now, let β be a hyperplane of R through α not containing R_1. Since $R_1 \subseteq A$, $A \cap R$ and β are two distinct hyperplanes of R. Hence, β is not contained in A. Since $\beta \subset R$, β satisfies the conditions (C1) and (C2) of Section 2 and the embedding $e := \tilde{e}/\beta$ is polarized, see Cardinali, De Bruyn and Pasini [3, Lemma 2.1]. Since $\beta \cap R_1 = \alpha \subseteq \langle \widetilde{e_1}(G)\rangle$, the hyperplane G arises from e_{F_1}. Since β is not contained in A, H does not arise from e. $\square$

There are examples known of fully embeddable thick dual polar spaces of rank at least 2 for which the absolutely universal embedding and the minimal full polarized embedding are not isomorphic:

(i) the Hermitian dual polar space $DH(2n-1,4)$, $n \geqslant 3$ (Li [15]; Cardinali, De Bruyn and Pasini [3, Theorem 4.1]);
(ii) the symplectic dual polar space $DW(2n-1,\mathbb{F})$, where $\mathbb{F}$ is a field whose characteristic is a prime p and $n \geqslant 2(p-1)$ (Cooperstein [6, Theorem B]; De Bruyn [10, Corollary 2.1]; De Bruyn and Pasini [11, Corollary 1.2]).

So, the situation mentioned in Proposition 8.2 can occur if Δ is a symplectic or Hermitian dual polar space of suitable rank over a suitable field.

References

[1] R.J. Blok, A.E. Brouwer, The geometry far from a residue, in: Groups and Geometries, Siena, 1996, in: Trends Math., Birkhäuser, Basel, 1998, pp. 29–38.
[2] P.J. Cameron, Dual polar spaces, Geom. Dedicata 12 (1982) 75–85.
[3] I. Cardinali, B. De Bruyn, A. Pasini, Minimal full polarized embeddings of dual polar spaces, J. Algebraic Combin. 25 (2007) 7–23.
[4] I. Cardinali, B. De Bruyn, A. Pasini, On the simple connectedness of hyperplane complements in dual polar spaces, Discrete Math. 309 (2009) 294–303.
[5] B.N. Cooperstein, On the generation of dual polar spaces of unitary type over finite fields, European J. Combin. 18 (1997) 849–856.
[6] B.N. Cooperstein, On the generation of dual polar spaces of symplectic type over finite fields, J. Combin. Theory Ser. A 83 (1998) 221–232.
[7] B. De Bruyn, Near Polygons, Front. Math., Birkhäuser Verlag, Basel, 2006.
[8] B. De Bruyn, On the Grassmann-embeddings of the Hermitian dual polar spaces, Linear Multilinear Algebra 56 (2008) 665–677.
[9] B. De Bruyn, On isometric full embeddings of symplectic dual polar spaces into Hermitian dual polar spaces, Linear Algebra Appl. 430 (2009) 2541–2552.
[10] B. De Bruyn, On the Grassmann modules for the symplectic groups, J. Algebra 324 (2010) 218–230.
[11] B. De Bruyn, A. Pasini, Generating symplectic and Hermitian dual polar spaces over arbitrary fields nonisomorphic to $\mathbb{F}_2$, Electron. J. Combin. 14 (2007), Research Paper 54, 17 pp.
[12] B. De Bruyn, P. Vandecasteele, Valuations and hyperplanes of dual polar spaces, J. Combin. Theory Ser. A 112 (2005) 194–211.
[13] P.M. Johnson, Semiquadratic sets and embedded polar spaces, J. Geom. 64 (1999) 102–127.
[14] A. Kasikova, E.E. Shult, Absolute embeddings of point-line geometries, J. Algebra 238 (2001) 265–291.
[15] P. Li, On the universal embedding of the $U_{2n}(2)$ dual polar space, J. Combin. Theory Ser. A 98 (2002) 235–252.
[16] J. McInroy, S. Shpectorov, On the simple connectedness of hyperplane complements in dual polar spaces, II, Discrete Math. 310 (2010) 1381–1388.
[17] A. Pasini, Diagram Geometries, Oxford Sci. Publ., Clarendon Press, Oxford University Press, New York, 1994.
[18] M.A. Ronan, Embeddings and hyperplanes of discrete geometries, European J. Combin. 8 (1987) 179–185.
[19] E.E. Shult, On Veldkamp lines, Bull. Belg. Math. Soc. Simon Stevin 4 (1997) 299–316.
[20] J. Tits, Buildings of Spherical Type and Finite BN-Pairs, Lecture Notes in Math., vol. 386, Springer, Berlin, 1974.
[21] F.D. Veldkamp, Polar Geometry I–IV, V, Indag. Math. 21–22 (1959) 512–551, and 207–212.